

AN OBSERVATION ON POSITIVE DEFINITE FORMS

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ABSTRACT. Given two positive definite forms $f, g \in \mathbb{R}[x_1, \dots, x_n]$, we prove that fg^r lies in the interior of the sums of squares cone for large r .

Fix $n \geq 1$ and write $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$. A form (homogeneous polynomial) $f \in \mathbb{R}[\mathbf{x}]$ is *positive definite* if $f(\xi) > 0$ for every $0 \neq \xi \in \mathbb{R}^n$. For every integer $d \geq 0$ let $\mathbb{R}[\mathbf{x}]_d$ denote the space of forms of degree d . Let $\Sigma_{2d} \subseteq \mathbb{R}[\mathbf{x}]_{2d}$ be the set of all forms of degree $2d$ that are *sos*, i.e. sums of squares of forms. It is well known that Σ_{2d} is a full-dimensional closed convex cone in $\mathbb{R}[\mathbf{x}]_{2d}$.

The main result of [1] implies (see [1] Remark 4.6):

Theorem 1. *Let f, g be positive definite forms in $\mathbb{R}[x_1, \dots, x_n]$, with g not constant. Then there is $r_0 \geq 0$ such that the form fg^r is a sum of squares for all $r \geq r_0$.*

A form f of degree $2d$ will be said to be a *strict sum of squares*, or *strictly sos*, if f lies in the interior of the cone Σ_{2d} . It is equivalent that f has a sum of squares representation $f = f_1^2 + \dots + f_N^2$ in which f_1, \dots, f_N form a linear basis of $\mathbb{R}[\mathbf{x}]_d$. The purpose of this note is to observe that Theorem 1 can be sharpened as follows:

Proposition 2. *Let f, g be positive definite forms in $\mathbb{R}[x_1, \dots, x_n]$, with g not constant. Then there is $r_0 \geq 0$ such that the form fg^r is a strict sum of squares for all $r \geq r_0$.*

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Before giving the proof we need a bit of preparation. Given a form $f \in \Sigma_{2d}$, let U_f be the set of forms $p \in \mathbb{R}[\mathbf{x}]_d$ for which there exists a real number $c > 0$ such that $f - cp^2$ is a sum of squares. Then U_f is a linear subspace of $\mathbb{R}[\mathbf{x}]_d$, and f is strictly sos if and only if $U_f = \mathbb{R}[\mathbf{x}]_d$. More generally, the faces of the cone Σ_{2d} are precisely the sets $F_U \Sigma_{2d} := \{f \in \Sigma_{2d} : U_f \subseteq U\}$, for U a subspace of $\mathbb{R}[\mathbf{x}]_d$. (We will not use this fact.) If U, V are linear subspaces of $\mathbb{R}[\mathbf{x}]$, let UV denote the linear subspace spanned by the products uv with $u \in U$ and $v \in V$. Let f and g be two forms that are sums of squares. Clearly we have $U_f U_g \subseteq U_{fg}$, and $U_f + U_g \subseteq U_{f+g}$ if $\deg(f) = \deg(g)$. In particular, the product of two forms that are strictly sos is again strictly sos.

Proof. By Theorem 1 there is $k \geq 0$ such that fg^k and fg^{k+1} are sos. Replacing f with fg^k or fg^{k+1} and g with g^2 , we can assume that both forms f and g are sos.

Let $\deg(f) = 2d$ and $\deg(g) = 2e$, write $A = \mathbb{R}[\mathbf{x}]$. Let q be a strictly sos form with $\deg(q) = 2e$ such that $g - q$ is positive definite, for example $q = c(x_1^2 + \dots + x_n^2)^e$

for suitable real $c > 0$. By Theorem 1 there exists $k \geq 1$ such that the forms $g^k(g-q)$ and $q^k(g-q)$ are both sos. Then for every $r \geq 2k$ the form

$$g^r - q^r = \sum_{j=0}^{r-1} (g-q)g^j q^{r-1-j}$$

is sos, since this is true for every summand on the right. Since q^r is strictly sos, this implies that g^r is strictly sos for every $r \geq 2k$.

Fix $p \in A_d$. There is a real number $c > 0$ for which $f - cp^2$ is positive definite. By Theorem 1 there is an integer $r(p) \geq 2k$ such that $(f - cp^2)g^r$ is sos for $r \geq r(p)$. This implies $U_{p^2}U_{g^r} \subseteq U_{fg^r}$ for these r , and therefore $pA_{re} \subseteq U_{fg^r}$ since g^r is strictly sos. Repeat this argument for every monomial p of degree d , and let r_0 be the maximum of the respective numbers $r(p)$. For every $r \geq r_0$ we then have $A_d A_{re} = A_{d+re} \subseteq U_{fg^r}$, which means that fg^r is strictly sos. \square

Remark. Stengle's Positivstellensatz says that a polynomial f has strictly positive values if and only if there exist sums of squares of polynomials g and h such that $fg = 1 + h$. For a form, being a strict sum of squares is a certificate for being positive definite. Therefore Proposition 2 can be seen as a strong homogeneous version of Stengle's Positivstellensatz, in which the multiplier can be chosen to be a power of any preassigned (nonconstant) positive definite form.

REFERENCES

- [1] C. Scheiderer: A Positivstellensatz for projective real varieties. *Manuscr. math.* **138**, 73–88 (2012).

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